Singular Integrals on Product Homogeneous Groups

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Singular integrals on product homogeneous groups

Yong Ding and Shuichi Sato

Abstract. We consider singular integral operators with rough kernels on the product space of homogeneous groups. We prove L^p boundedness of them for $p \in (1, \infty)$ under a sharp integrability condition of the kernels.

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1. Introduction

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, where $d \geq 2$. We assume that $\mathbb{R}^d$ is equipped with multiplication given by a polynomial mapping which makes $\mathbb{R}^d$ a homogeneous group. This requires the existence of a dilation family $\{A_t\}_{t>0}$ on $\mathbb{R}^d$ such that each A_t is an automorphism of the group structure, where A_t is of the form

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_d} x_d), \quad x = (x_1, \dots, x_d),$$

with some real numbers $a_1, \dots, a_d$ satisfying $0 < a_1 \leq a_2 \leq \cdots \leq a_d$ (see [12], [25] and [13, Section 2 of Chapter 1]). We also write $\mathbb{R}^d = \mathbb{H}$. Therefore, in addition to the Euclidean structure, $\mathbb{H}$ is equipped with a homogeneous nilpotent Lie group structure, where Lebesgue measure is bi-invariant Haar measure, the identity is the origin 0 and $x^{-1} = -x$. We note that multiplication xy, $x, y \in \mathbb{H}$, satisfies

(1) $A_t(xy) = (A_t x)(A_t y)$, $x, y \in \mathbb{H}$, $t > 0$;
(2) $(ux)(vx) = ux + vx$, $x \in \mathbb{H}$, $u, v \in \mathbb{R}$;
(3) if $z = xy$, $z_k = P_k(x, y)$, then $P_1(x, y) = x_1 + y_1$ and $P_k(x, y) = x_k + y_k + R_k(x, y)$ for $k \geq 2$, where $R_k(x, y)$ is a polynomial depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$.

We have a norm function $r(x)$ satisfying $r(A_t x) = tr(x)$ for $t > 0$ and $x \in \mathbb{R}^d$. We may assume the following:

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(4) the function r is continuous on $\mathbb{R}^d$ and smooth in $\mathbb{R}^d \setminus \{0\}$;
(5) $r(x+y) \le B_1(r(x)+r(y))$, $r(xy) \le B_2(r(x)+r(y))$ for some constants $B_1, B_2 \ge 1$;
(6) $r(x^{-1}) = r(x)$ and if we denote by $|x|$ the Euclidean norm for $x \in \mathbb{R}^d$, then

$$c_1|x|^{\alpha_1} \le r(x) \le c_2|x|^{\alpha_2} \quad \text{if } r(x) \ge 1,$$
$$c_3|x|^{\beta_1} \le r(x) \le c_4|x|^{\beta_2} \quad \text{if } r(x) \le 1$$

for some positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2;
(7) if we define $\Sigma_d = \{x \in \mathbb{R}^d : r(x) = 1\}$, Σ_d coincides with $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$;
(8) if $\gamma = a_1 + \cdots + a_d$ (the homogeneous dimension of $\mathbb{H}$), then $dx = t^{\gamma-1}\, dS_d\, dt$, that is,

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \int_{\Sigma_d} f(A_t\theta) t^{\gamma-1}\, dS_d(\theta)\, dt$$

with $dS_d = \omega\, d\sigma_d$, where ω is a strictly positive C^∞ function on Σ_d and $d\sigma_d$ is the Lebesgue surface measure on Σ_d.

If we define a left invariant quasi-metric d by $d(x,y) = r(x^{-1}y)$, the space $\mathbb{H}$ with the quasi-metric d can be regarded as a space of homogeneous type. See [3, 7, 12, 13, 16, 23, 24, 25] for more details about background materials.

The convolution $f * g$ on $\mathbb{H}$ is defined by

$$f * g(x) = \int_{\mathbb{H}} f(y) g(y^{-1}x)\, dy.$$

Let Ω be locally integrable in $\mathbb{R}^d \setminus \{0\}$. We assume that Ω is homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \ne 0$, $t > 0$ and that

$$\int_{\Sigma_d} \Omega(\theta)\, dS_d(\theta) = 0.$$

We define the singular integral

$$Tf(x) = \text{p.v.} f * K(x) = \text{p.v.} \int_{\mathbb{R}^d} f(y) K(y^{-1}x)\, dy \tag{1.1}$$

for appropriate functions f, where $K(x) = \Omega(x') r(x)^{-\gamma}$, $x' = A_{r(x)^{-1}} x$ for $x \ne 0$. The following result was proved in [25].

Theorem A. *Let Tf be as in* (1.1). *Suppose that $\Omega \in L \log L(\Sigma_d)$. Then, T is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.*

The maximal singular integral operator is defined by

$$T_* f(x) = \sup_{\epsilon > 0} \left| \int_{r(y) > \epsilon} f(xy^{-1}) K(y)\, dy \right|. \tag{1.2}$$

Then the following result is known (see [21]).

Theorem B. *Suppose that $\Omega \in L \log L(\Sigma_d)$. Let $T_* f$ be defined as in* (1.2). *Then, T_* is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$.*

See [4, 5, 6, 8, 14, 15, 16, 18] for relevant results and also [19, 22, 25] for weak $(1,1)$ boundedness.

An analogue of a theory of Duoandikoetxea and Rubio de Francia [10] for homogeneous groups was developed in [21], where the use of Fourier transform estimates was replaced by a variant of the L^2 estimates given in T. Tao [25]. The theory enables us to prove Theorem B and to give a different proof of Theorem A. In this note we shall show that the theory extends to the case of product spaces of homogeneous groups. Consequently, we can obtain an analogue of Theorem A for multiple singular integrals with rough kernels.

We consider the product space $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $n = n_1 + n_2$ and $\mathbb{R}^{n_1} = \mathbb{H}_1$, $\mathbb{R}^{n_2} = \mathbb{H}_2$ are homogeneous groups with dilations $A_t^{(1)}$, $A_t^{(2)}$ and norm functions r_1, r_2, respectively. Let $\Omega \in L^1(\Sigma_{n_1} \times \Sigma_{n_2})$ satisfy

$$\int_{\Sigma_{n_1}} \Omega(u,v)\, dS_{n_1}(u) = \int_{\Sigma_{n_2}} \Omega(u,v)\, dS_{n_2}(v) = 0 \tag{1.3}$$

for all $(u,v) \in \Sigma_{n_1} \times \Sigma_{n_2}$. Define the singular integral

$$Tf(x,y) = \text{p.v.}\, f * K(x,y) = \text{p.v.} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} f(xu^{-1}, yv^{-1}) K(u,v)\, du\, dv, \tag{1.4}$$

where $K(u,v) = r_1(u)^{-\gamma_1} r_2(v)^{-\gamma_2} \Omega(u', v')$, $u' = A^{(1)}_{r_1(u)^{-1}} u$, $v' = A^{(2)}_{r_2(v)^{-1}} v$, with γ_1 and γ_2 denoting the homogeneous dimensions of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Then we shall prove the following.

Theorem 1. *Suppose that* $\Omega \in L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$. *Let* T *be as in* (1.4). *Then,* T *is bounded on* $L^p(\mathbb{H}_1 \times \mathbb{H}_2)$ *for all* $p \in (1, \infty)$.

Also, we consider the maximal singular integral

$$T_* f(x,y) = \sup_{\substack{\epsilon_1 > 0, \\ \epsilon_2 > 0}} \left| \int_{\substack{r_1(u) > \epsilon_1, \\ r_2(v) > \epsilon_2}} f(xu^{-1}, yv^{-1}) K(u,v)\, du\, dv \right|. \tag{1.5}$$

We anticipate L^p boundedness of T_* under a condition of the kernels similar to the one in Theorem 1.

See [1, 2, 9, 11] for previous works about singular integrals on product of Euclidean spaces. Our methods will give different proofs for some previous results, where singular integrals are defined by Euclidean convolution, since our proof of Theorem 1 will not use Fourier transform estimates explicitly. Theorem 1 is an extension of a result of [2] to the case of singular integrals on product homogeneous groups. The optimality of the kernel class $L(\log L)^2$, in the case of Euclidean convolution, can be found in [2].

To prove Theorem 1, we apply extrapolation arguments via the following estimate.

Proposition 1. *Let* $1 < p < \infty$, $1 < s \le 2$ *and* $\Omega \in L^s(\Sigma_{n_1} \times \Sigma_{n_2})$. *Then, there exists a constant* C_p *independent of* s *and* Ω *such that*

$$\|Tf\|_p \le C_p (s-1)^{-2} \|\Omega\|_s \|f\|_p.$$

We can prove Theorem 1 from Proposition 1 by decomposing $\Omega \in L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$ as $\Omega = \sum_{k=1}^{\infty} c_k \Omega_k$, where each Ω_k satisfies (1.3), $\sup_{k\geq 1} \|\Omega_k\|_{1+1/k} \leq 1$ and $\{c_k\}$ is a sequence of non-negative real numbers such that $\sum_{k=1}^{\infty} k^2 c_k < \infty$ (see [20] and also [17, 18]).

In Section 2 we prove a basic L^2-estimate (Lemma 1) by applying methods of Tao [25]. This enables us to adapt the theory of [10] for the case of multiple singular integrals on product homogeneous groups to prove Proposition 1 in Section 3. In Section 4 we prove an estimate for a certain maximal function which is closely related to the maximal singular integral in (1.5). In the proofs of the results, we deal with real valued functions only to simplify our arguments. In this note, the letter C, along with some others, will be used to denote non-negative constants which may be different in different occurrences.

2. Orthogonality estimates in L^2 via convolution

We write $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^n$, $x^{(1)} \in \mathbb{R}^{n_1}$, $x^{(2)} \in \mathbb{R}^{n_2}$, $n = n_1 + n_2$. Let $B_i(x^{(i)}, t)$ be the ball with center $x^{(i)}$ and radius t in $\mathbb{R}^{n_i}$ defined by

$$B_i(x^{(i)}, t) = \{y^{(i)} \in \mathbb{R}^{n_i} : r_i((x^{(i)})^{-1} y^{(i)}) < t\}.$$

Let $\phi^{(i)}$, $i = 1, 2$, be a C^∞ function on $\mathbb{R}^{n_i}$ with support in $B_i(0,1) \setminus B_i(0,1/2)$. We assume that $\int \phi^{(i)} = 1$, $\phi^{(i)} = \tilde{\phi}^{(i)}$, $\phi^{(i)}(x^{(i)}) \geq 0$ for all $x^{(i)} \in \mathbb{R}^{n_i}$, where $\tilde{\phi}^{(i)}(x^{(i)}) = \phi^{(i)}((x^{(i)})^{-1})$. Set

$$\Delta_k^{(i)} = \delta_{\rho^{k-1}}^{(i)} \phi^{(i)} - \delta_{\rho^k}^{(i)} \phi^{(i)}, \quad k \in \mathbb{Z},$$

where $\delta_t^{(i)} \phi^{(i)}(x^{(i)}) = t^{-\gamma_i} \phi^{(i)}((A_t^{(i)})^{-1} x^{(i)})$, $\rho \geq 2$ and $\mathbb{Z}$ denotes the set of integers. Define $\delta_{s,t} = \delta_s^{(1)} \otimes \delta_t^{(2)}$. Note that $\operatorname{supp}(\Delta_k^{(i)}) \subset B_i(0, \rho^k) \setminus B_i(0, \rho^{k-1}/2)$, $\Delta_k^{(i)} = \tilde{\Delta}_k^{(i)}$ and $\sum_k \Delta_k^{(i)} = \delta^{(i)}$, where $\delta^{(i)}$ is the delta function on $\mathbb{R}^{n_i}$. Choose $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, satisfying

$$\operatorname{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$
$$(\log 2) \sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$
$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \ldots,$$

where c_m is a constant independent of ρ; this is possible since we assume $\rho \geq 2$. We may assume that $\psi_j(t) = \psi_0(\rho^{-j} t)$.

Define

$$S_{j,k} F(x) = \int_0^\infty \int_0^\infty \psi_j(s) \psi_k(t) \delta_{s,t} F(x) \, \frac{ds}{s} \, \frac{dt}{t}, \tag{2.1}$$

where $F \in L^1(\mathbb{R}^n)$, $\operatorname{supp}(F) \subset D_0$, $D_0 = D_0^{(1)} \times D_0^{(2)}$, $D_0^{(i)} = \{x^{(i)} \in \mathbb{R}^{n_i} : 1 \leq r_i(x^{(i)}) \leq 2\}$. Let $K_0(x) = K(x) \chi_{D_0}(x)$, where χ_E denotes the characteristic function of a set E. Then $\sum_{j,k \in \mathbb{Z}} S_{j,k} K_0 = K$.

Let $\Phi^{(i)}$ be a non-negative smooth function on $\mathbb{R}^{n_i}$ such that $\int \Phi^{(i)}(x^{(i)})\,dx^{(i)} = 1$. We also assume that $\tilde{\Phi}^{(i)} = \Phi^{(i)}$, $\operatorname{supp}(\Phi^{(i)}) \subset B_i(0,1) \setminus B_i(0,1/2)$. For $F \in L^1(\mathbb{R}^n)$ with $\operatorname{supp}(F) \subset D_0$, define the operator $U_\sigma = U_\sigma(F)$ by

$$U_\sigma f = U_\sigma(F)(f) = \sum_{j,k} \sigma_{j,k} f * \nu_{j,k}, \tag{2.2}$$

where

$$\nu_{j,k}(x) = \nu_{j,k}(F)(x) = S_{j,k}F(x) - \Phi^{(1)}_{j,k}(x) - \Phi^{(2)}_{j,k}(x) + \Phi_{j,k}(x),$$

$$\Phi^{(1)}_{j,k}(x) = \Phi^{(1)}_{j,k}(F)(x) = \left(\int S_{j,k}F(x)\,dx^{(1)}\right)\delta^{(1)}_{\rho^j}\Phi^{(1)}(x^{(1)}),$$

$$\Phi^{(2)}_{j,k}(x) = \Phi^{(2)}_{j,k}(F)(x) = \left(\int S_{j,k}F(x)\,dx^{(2)}\right)\delta^{(2)}_{\rho^k}\Phi^{(2)}(x^{(2)}),$$

$$\Phi_{j,k}(x) = \Phi_{j,k}(F)(x) = \left(\int S_{j,k}F(x)\,dx\right)\delta_{\rho^j,\rho^k}\Phi(x), \quad \Phi = \Phi^{(1)} \otimes \Phi^{(2)},$$

and $\sigma = \{\sigma_{j,k}\}$ is an arbitrary sequence such that $\sigma_{j,k} = 1$ or -1. We note that $\int \nu_{j,k}(x)\,dx^{(i)} = 0$, $S_{j,k}K_0 = \nu_{j,k}(K_0)$ and $U_\sigma(K_0)(f) = Tf$ if $\sigma_{j,k} = 1$ for all j,k. For $s \geq 1$, let $L^s(D_0)$ denote the subspace of $L^s(\mathbb{H}_1 \times \mathbb{H}_2)$ consisting of functions F supported in D_0. We prove the following L^2 estimates.

Lemma 1. *Suppose that $F \in L^s(D_0)$, $s \in (1,2]$. Let $\nu_{j_1,j_2} = \nu_{j_1,j_2}(F)$, $a(t) = \min(1, \rho^{-t})$. Then, for $j_i, k_i \in \mathbb{Z}$, $i = 1,2$, we have*

$$\|f * \nu_{j_1,j_2} * \Delta_{k_1,k_2}\|_2 \leq C(\log\rho)^2 \left(\prod_{i=1}^{2} a(\epsilon(|j_i - k_i| - c)/s')\right) \|F\|_s \|f\|_2 \tag{2.3}$$

for some positive constants C, ϵ and c independent of ρ, s and F, where $\Delta_{k_1,k_2} = \Delta^{(1)}_{k_1} \otimes \Delta^{(2)}_{k_2}$ and $s' = s/(s-1)$.

Proof. It suffices to prove Lemma 1 by assuming $j_i = 0$. This can be seen by applying $\delta_{\rho^{-j_1},\rho^{-j_2}}$ to $f * \nu_{j_1,j_2} * \Delta_{k_1,k_2}$ and noting that $\delta_{s,t}(f * g) = (\delta_{s,t}f) * (\delta_{s,t}g)$, $\delta_{\rho^{-j_1},\rho^{-j_2}}\Delta_{k_1,k_2} = \Delta_{k_1-j_1,k_2-j_2}$.

Let $\nu = \nu_{0,0}$. If $k_1, k_2 \geq 0$, then from the cancellation condition for ν and the smoothness of Δ_{k_1,k_2} we shall show

$$\|\nu * \Delta_{k_1,k_2}\|_1 \leq C(\log\rho)^2 \|F\|_1 \prod_{i=1}^{2} a(\epsilon k_i - \tau) \tag{2.4}$$

for some $\epsilon, \tau > 0$, which implies the conclusion by Young's inequality. We need the following estimates.

Lemma 2. *Suppose that $F \in L^q(D_0)$ for some $q \in [1,2]$. Put $S = S_{0,0}F$. Then*

$$\|S\|_q \leq C(\log\rho)^2 \|F\|_q,$$

where the constant C is independent of ρ, q and F.

Proof. Since $\int_0^\infty \psi_0(t)\,dt/t \le (\log 2)^{-1}2\log\rho$, by Hölder's inequality it follows that

$$\begin{aligned}
\|S\|_q^q &\le ((\log 2)^{-1}2\log\rho)^{2q/q'} \iiint \psi_0(s)\psi_0(t)|\delta_{s,t}F(x)|^q \,\frac{ds}{s}\,\frac{dt}{t}\,dx \\
&= ((\log 2)^{-1}2\log\rho)^{2q/q'} \iiint \psi_0(s)\psi_0(t)s^{\gamma_1(1-q)}t^{\gamma_2(1-q)}\,\frac{ds}{s}\,\frac{dt}{t}|F(x)|^q\,dx \\
&\le C(\log\rho)^{2q/q'+2}\|F\|_q^q.
\end{aligned}$$

This implies the conclusion. □

To show (2.4), we first note that

$$\|\nu * \Delta_{k_1,k_2}\|_1 \le C(\log\rho)^2\|F\|_1 \tag{2.5}$$

by Lemma 2 with $q = 1$. Let $t_1 = \rho^{k_1-1}, t_2 = \rho^{k_2-1}$. Then $\Delta_{k_1,k_2} = \delta_{t_1,t_2}\Delta_{1,1}$. Since

$$\int \nu\,dy^{(1)} = \int \nu\,dy^{(2)} = 0,$$

we have

$$\begin{aligned}
&\nu * \Delta_{k_1,k_2}(x) \\
&= \int \prod_{i=1}^{2} t_i^{-\gamma_i}\left(\Delta_1^{(i)}((A_{t_i}^{(i)})^{-1}((y^{(i)})^{-1}x^{(i)})) - \Delta_1^{(i)}((A_{t_i}^{(i)})^{-1}x^{(i)})\right)\nu(y)\,dy \\
&= \int t_1^{-\gamma_1}\left(\Delta_1^{(1)}((A_{t_1}^{(1)})^{-1}((y^{(1)})^{-1}x^{(1)})) - \Delta_1^{(1)}((A_{t_1}^{(1)})^{-1}x^{(1)})\right) \\
&\qquad\qquad \times \Delta_{k_2}^{(2)}((y^{(2)})^{-1}x^{(2)})\nu(y)\,dy. \\
&= \int t_2^{-\gamma_2}\left(\Delta_1^{(2)}((A_{t_2}^{(2)})^{-1}((y^{(2)})^{-1}x^{(2)})) - \Delta_1^{(2)}((A_{t_2}^{(2)})^{-1}x^{(2)})\right) \\
&\qquad\qquad \times \Delta_{k_1}^{(1)}((y^{(1)})^{-1}x^{(1)})\nu(y)\,dy.
\end{aligned}$$

Therefore, if $k_1, k_2 \ge 1$, then it is not difficult to see that

$$\|\nu * \Delta_{k_1,k_2}\|_1 \le C\|F\|_1\rho^{-k_1\epsilon-k_2\epsilon+\tau} \tag{2.6}$$

for some $\epsilon, \tau > 0$; also, if $k_1 \ge 1$,

$$\|\nu * \Delta_{k_1,k_2}\|_1 \le C\rho^{-k_1\epsilon+\tau}\|F\|_1, \tag{2.7}$$

and if $k_2 \ge 1$,

$$\|\nu * \Delta_{k_1,k_2}\|_1 \le C\rho^{-k_2\epsilon+\tau}\|F\|_1 \tag{2.8}$$

(see the proof of (3.7) of [21]). By (2.5)–(2.8), we have (2.4) for $k_1, k_2 \ge 0$.

Let

$$S_j^{(i)}F(x^{(i)}) = \int_0^\infty \psi_j(s)\delta_s^{(i)}F(x^{(i)})\,\frac{ds}{s}.$$

Then $\Phi_{0,0}^{(1)}(x) = S_0^{(2)}G(x^{(2)})\Phi^{(1)}(x^{(1)})$, where

$$G(x^{(2)}) = \int S_0^{(1)}F(x)\,dx^{(1)}$$

with $S_0^{(1)}$ acting on $x^{(1)}$ variable only. Note that

$$f*\Phi_{0,0}^{(1)}*\Delta_{k_1,k_2} = f*(\Phi^{(1)}\otimes\delta^{(2)})*(\Delta_{k_1}^{(1)}\otimes\delta^{(2)})*(\delta^{(1)}\otimes S_0^{(2)}G)*(\delta^{(1)}\otimes\Delta_{k_2}^{(2)}).$$

By Lemma 1 of [21]

$$\|g*(\delta^{(1)}\otimes S_0^{(2)}G)*(\delta^{(1)}\otimes\Delta_{k_2}^{(2)})\|_2 \le C(\log\rho)a(\epsilon(|k_2|-c)/s')\|G\|_s\|g\|_2.$$

It is easy to see that

$$\|G\|_s \le C(\log\rho)\|F\|_s.$$

Combining these estimates, we have

$$\begin{aligned}&\|f*\Phi_{0,0}^{(1)}*\Delta_{k_1,k_2}\|_2 \\ &\le C(\log\rho)^2a(\epsilon(|k_2|-c)/s')\|F\|_s\|f*(\Phi^{(1)}\otimes\delta^{(2)})*(\Delta_{k_1}^{(1)}\otimes\delta^{(2)})\|_2 \\ &\le C(\log\rho)^2a(\epsilon(|k_2|-c)/s')\|F\|_s\|f\|_2.\end{aligned} \tag{2.9}$$

If $k_1\le -1$, since $\int\Delta_0^{(1)}=0$, as in the proof of (2.6) we have

$$\|\Phi^{(1)}*\Delta_{k_1}^{(1)}\|_1 = \|(\delta_{\rho^{-k_1}}^{(1)}\Phi^{(1)})*\Delta_0^{(1)}\|_1 \le Ca(\epsilon(|k_1|-c)).$$

From this, (2.9) and Young's inequality, it follows that

$$\|f*\Phi_{0,0}^{(i)}*\Delta_{k_1,k_2}\|_2 \le C\|F\|_s\|f\|_2(\log\rho)^2\prod_{j=1}^{2}a(\epsilon(|k_j|-c)/s') \tag{2.10}$$

if $k_i\le -1$, for $i=1$; the result for $i=2$ can be proved similarly.

If $k_1,k_2\le -1$, since $\int\Delta_0^{(i)}=0$ and $\|\Phi*\Delta_{k_1,k_2}\|_1 = \|\delta_{\rho^{-k_1},\rho^{-k_1}}\Phi*\Delta_{0,0}\|_1$, by the proof of (2.6) and Lemma 2 we have

$$\|f*\Phi_{0,0}*\Delta_{k_1,k_2}\|_2 \le C\|F\|_1\|f\|_2(\log\rho)^2\prod_{i=1}^{2}a(\epsilon(|k_i|-c)). \tag{2.11}$$

If $k_1\ge 0$ and $k_2\le -1$, we write

$$\Phi_{0,0}*\Delta_{k_1,k_2} = (\Phi_{0,0}*\Delta_{k_1,k_2}-\Phi_{0,0}^{(2)}*\Delta_{k_1,k_2})+\Phi_{0,0}^{(2)}*\Delta_{k_1,k_2}.$$

We note that the first term on the right hand side is equal to

$$\left(\left((\int S\,dx)\Phi^{(1)}-\int S\,dx^{(2)}\right)*\Delta_{k_1}^{(1)}\right)\otimes\left(\Phi^{(2)}*\Delta_{k_2}^{(2)}\right),$$

where $S=S_{0,0}F$ as above. Therefore, arguing as above, we see that

$$\|\Phi_{0,0}*\Delta_{k_1,k_2}-\Phi_{0,0}^{(2)}*\Delta_{k_1,k_2}\|_1 \le C\|F\|_1\prod_{i=1}\ 2\rho^{-\epsilon(|k_i|-c)}.$$

Combining this with Lemma 2, we have

$$\|\Phi_{0,0}*\Delta_{k_1,k_2}-\Phi_{0,0}^{(2)}*\Delta_{k_1,k_2}\|_1 \le C\|F\|_1\|f\|_2(\log\rho)^2\prod_{i=1}^{2}a(\epsilon(|k_i|-c)).$$

By this and (2.10) with $i=2$, we have (2.11) in the case $k_1 \ge 0$ and $k_2 \le -1$. The case $k_1 \le -1$ and $k_2 \ge 0$ can be handled similarly.

We shall prove

$$\|f * S * \Delta_{k_1,k_2}\|_2 \le C\|F\|_s\|f\|_2(\log\rho)^2 \prod_{i=1}^{2} a(\epsilon(|k_i|-c)/s') \tag{2.12}$$

for $k_1, k_2 \le -1$,

$$\|f * S * \Delta_{k_1,k_2}\|_2 \le C\|F\|_s(\log\rho)^2 a(\epsilon(|k_2|-c)/s')\|f\|_2 \tag{2.13}$$

for $k_1 \ge 0, k_2 \le -1$,

$$\|f * S * \Delta_{k_1,k_2}\|_2 \le C\|F\|_s(\log\rho)^2 a(\epsilon(|k_1|-c)/s')\|f\|_2 \tag{2.14}$$

for $k_1 \le -1, k_2 \ge 0$.

By (2.10), (2.11) and (2.12), we get the conclusion of Lemma 1 for $k_1, k_2 \le -1$. If $k_1 \ge 0, k_2 \le -1$, by (2.9), (2.10) with $i=2$, (2.11) and (2.13) we have

$$\|f * \nu * \Delta_{k_1,k_2}\|_2 \le C\|F\|_s(\log\rho)^2 a(\epsilon(|k_2|-c)/s')\|f\|_2.$$

Combining this with (2.7) and Lemma 2, we get the desired inequality of Lemma 1 for $k_1 \ge 0, k_2 \le -1$. If $k_1 \le -1, k_2 \ge 0$, by an analogue of (2.9) for $\|f * \Phi^{(2)}_{0,0} * \Delta_{k_1,k_2}\|_2$, (2.10) with $i=1$, (2.11) and (2.14) we have

$$\|f * \nu * \Delta_{k_1,k_2}\|_2 \le C\|F\|_s(\log\rho)^2 a(\epsilon(|k_1|-c)/s')\|f\|_2.$$

By this and (2.8) with Lemma 2 we get the conclusion of Lemma 1 for $k_1 \le -1, k_2 \ge 0$. If $k_1, k_2 \ge 0$, we can use (2.4) to prove the estimate in Lemma 1. This will complete the proof of Lemma 1.

It remains to prove (2.12), (2.13) and (2.14). Let $k_1, k_2 \le -1$. On account of Lemma 2 and the T^*T method, to prove (2.12) it suffices to show that

$$\begin{aligned}\Big\| f * \Big(\Delta_{k_1,k_2} * \tilde{S} * S * \Delta_{k_1,k_2}\Big)_*^{n_1} \Big\|_2 \\ \le C(\log\rho)^{4n_1}\rho^{\epsilon(k_1+k_2+c)/s'}\|F\|_s^{2n_1}\|f\|_2,\end{aligned} \tag{2.15}$$

for some $\epsilon, c > 0$, where f_*^m denotes the convolution product of m factors of f and we may assume $n_1 \ge n_2$ without loss of generality. This is deduced from Young's inequality and the L^1 estimate

$$\Big\| \Big(\Delta_{k_1,k_2} * \tilde{S} * S * \Delta_{k_1,k_2}\Big)_*^{n_1} \Big\|_1 \le C(\log\rho)^{4n_1}\rho^{\epsilon(k_1+k_2+c)/s'}\|F\|_s^{2n_1}. \tag{2.16}$$

Note that $\|\Delta'_{k_1,k_2} * \tilde{S}\|_1 \le C(\log\rho)^2\|F\|_1$ and $\Delta'_{k_1,k_2} * \tilde{S}(x) = \int \delta_y(x)\Delta'_{k_1,k_2} * \tilde{S}(y)\,dy$, where $\delta_y(x)$ is the delta function concentrated at y and Δ'_{k_1,k_2} is either Δ_{k_1,k_2} or $\Delta_{k_1,k_2} * \Delta_{k_1,k_2}$. Therefore, we get (2.16) if we prove

$$\Big\|\delta_{w_1} * S * \cdots * \delta_{w_{n_1}} * S * \Delta_{k_1,k_2}\Big\|_1 \le C(\log\rho)^{2n_1}\rho^{\epsilon(k_1+k_2+c)/s'}\|F\|_s^{n_1}$$

uniformly for $w_1, \ldots, w_{n_1} \in B_1(0, C\rho^2) \times B_2(0, C\rho^2)$, $w_j = (w_j^{(1)}, w_j^{(2)})$. This is a consequence of

$$\left|\left\langle \delta_{w_1} * S * \cdots * \delta_{w_{n_1}} * S * \Delta_{k_1,k_2}, g\right\rangle\right| \leq C(\log\rho)^{2n_1}\rho^{\epsilon(k_1+k_2+c)/s'}\|F\|_s^{n_1} \tag{2.17}$$

uniformly in $w_1, \ldots, w_{n_1} \in B_1(0, C\rho^2) \times B_2(0, C\rho^2)$, for all g in C^∞ with compact support such that $\|g\|_\infty \leq 1$.

The inner product on the left hand side of (2.17) can be written as

$$\iiint \Delta_{k_1,k_2}(x) g \prod_{u=1}^{n_1}\left(F(y_u^{(1)}, y_u^{(2)})\prod_{i=1}^{2}\psi(t_u^{(i)})\right) dy\, \bar{d}t\, dx,$$

where $\psi(s) = \psi_0(s)$, $t^{(i)} = (t_1^{(i)}, \ldots, t_{n_1}^{(i)})$, $y^{(i)} = (y_1^{(i)}, \ldots, y_{n_1}^{(i)}) \in (D_0^{(i)})^{n_1}$, $\bar{d}t = \bar{d}t^{(1)}\bar{d}t^{(2)}$, $\bar{d}t^{(i)} = (dt_1^{(i)}/t_1^{(i)}) \ldots (dt_{n_1}^{(i)}/t_{n_1}^{(i)})$, $dy = dy^{(1)}dy^{(2)}$, $dy^{(i)} = dy_1^{(i)} \ldots dy_{n_1}^{(i)}$ and

$$g = g(H_1(y^{(1)}, t^{(1)})x^{(1)}, H_2(y^{(2)}, t^{(2)})x^{(2)}),$$

$$H_i(y^{(i)}, t^{(i)}) = \prod_{j=1}^{n_1} w_j^{(i)} A_{t_j^{(i)}}^{(i)} y_j^{(i)} = w_1^{(i)} A_{t_1^{(i)}}^{(i)} y_1^{(i)} \ldots w_{n_1}^{(i)} A_{t_{n_1}^{(i)}}^{(i)} y_{n_1}^{(i)}.$$

Let $DH_i(y^{(i)}, t^{(i)})$ be the $n_i \times n_i$ matrix whose jth column vector is $\partial_{t_j^{(i)}}^L H_i(y^{(i)}, t^{(i)})$, $1 \leq j \leq n_i$:

$$DH_i(y^{(i)}, t^{(i)}) = \left(\partial_{t_1^{(i)}}^L H_i(y^{(i)}, t^{(i)}), \ldots, \partial_{t_{n_i}^{(i)}}^L H_i(y^{(i)}, t^{(i)})\right),$$

where $\partial_{t_j^{(i)}}^L H_i$ is the left invariant derivative (see [25], [21]). Then, to obtain (2.17) it suffices to show that

$$\left|\iiint \Delta_{k_1,k_2}(x) g G_{ij} \prod_{u=1}^{n_1}\left(F(y_u^{(1)}, y_u^{(2)})\prod_{i=1}^{2}\psi(t_u^{(i)})\right) dy\, \bar{d}t\, dx\right| \leq C(\log\rho)^{2n_1}\rho^{\delta\epsilon(k_1+k_2+c)/s'}\|F\|_s^{n_1}, \tag{2.18}$$

for $i, j = 1, 2$, with $G_{ij} = G_{ij}(y^{(1)}, y^{(2)}, t^{(1)}, t^{(2)})$ denoting

$$\zeta_i\left(\rho^{-\epsilon k_1}\det(DH_1(y^{(1)}, t^{(1)}))\right)\zeta_j\left(\rho^{-\epsilon k_2}\det(DH_2(y^{(2)}, t^{(2)}))\right),$$

where ζ_1 is a function in $C_0^\infty(\mathbb{R})$ such that $0 \leq \zeta_1 \leq 1$, $\mathrm{supp}(\zeta_1) \subset [-1, 1]$, $\zeta_1(t) = 1$ for $t \in [-1/2, 1/2]$, $\zeta_2 = 1 - \zeta_1$, and δ, ϵ are small positive numbers. We prove (2.18) by considering three cases.

Case 1: $i = 1$ and $j = 1$. We note that

$$\int_{(D_0^{(i)})^{n_1}} \chi_{[0,1]}\left(\rho^{-k_i\epsilon}\left|\det(DH_i(y^{(i)}, t^{(i)}))\right|\right) dy^{(i)} \leq C\rho^{\delta\epsilon(k_i+c)} \tag{2.19}$$

uniformly in $t^{(i)} \in [1,\rho^2]^{n_1}$ and $w_1^{(i)},\dots,w_{n_1}^{(i)} \in B_i(0,C\rho^2)$. If $i=1$, or $i=2$ and $n_2=n_1$, this was proved in Section 3 of [21]. To prove it for the case $i=2$ and $n_2<n_1$, we note that the arguments of [21] easily implies that

$$\int_{(D_0^{(2)})^{n_2}} \chi_{[0,1]}\left(\rho^{-k_2\epsilon}\left|\det(DH_2(y^{(2)},t^{(2)}))\right|\right) dy_1^{(2)}\dots dy_{n_2}^{(2)} \le C\rho^{\delta\epsilon(k_2+c)} \tag{2.20}$$

uniformly in $t^{(2)} \in [1,\rho^2]^{n_1}$, $w_1^{(2)},\dots,w_{n_1}^{(2)} \in B_2(0,C\rho^2)$ and $y_{n_2+1}^{(2)},\dots,y_{n_1}^{(2)} \in (D_0^{(2)})^{n_1-n_2}$. Thus, integrating (2.20) with respect to $y_{n_2+1}^{(2)},\dots,y_{n_1}^{(2)}$, we obtain (2.19) for $i=2$. By (2.19), applying Hölder's inequality, we obtain (2.18) for $i,j=1$.

Case 2: $i=2$ and $j=2$. It suffices to prove

$$\left|\iint \Delta_{k_1,k_2}(x) g G_{22} \prod_{i=1}^{2}\prod_{u=1}^{n_1}\psi(t_u^{(i)})\,\bar{d}t\,dx\right| \le C\rho^{\delta\epsilon(k_1+k_2+c)} \tag{2.21}$$

uniformly in $y^{(i)} \in (D_0^{(i)})^{n_1}$, $i=1,2$, and $w_1,\dots,w_{n_1} \in B_1(0,C\rho^2)\times B_2(0,C\rho^2)$. To prove (2.21) we need the following four lemmas.

Lemma 3. *Let f be a continuous function on $\mathbb{R}^{n_i}$ such that*

$$\operatorname{supp}(f) \subset B_i(0,C_1), \quad \int f(x^{(i)})\,dx^{(i)} = 0, \quad \|f\|_1 \le C_2.$$

Then we can find functions $f_1,f_2,\dots,f_{n_i}$ such that

$$f(x^{(i)}) = \sum_{j=1}^{n_i} \partial_{x_j^{(i)}} f_j(x^{(i)}),$$

$$\operatorname{supp}(f_j) \subset B_i(0,C_1'), \quad \|f_j\|_1 \le C_2' \quad \textit{for } j=1,2,\dots,n_i.$$

Lemma 4. *There exist functions $F_j^{(i)}$ on $\mathbb{R}^{n_i}$, $j=1,2,\dots,n_i$, such that $\operatorname{supp}(F_j^{(i)}) \subset B_i(0,C\rho^{k_i})$, $\|F_j^{(i)}\|_1 \le C\rho^{k_i\alpha}$ for some $\alpha>0$ and*

$$\Delta_{k_i}^{(i)}(x^{(i)}) = \sum_{j=1}^{n_i} \partial_{x_j^{(i)}} F_j^{(i)}(x^{(i)}).$$

Lemma 5. *Define $DH_i(y^{(i)},t^{(i)})x^{(i)}$ in the same way as $DH_i(y^{(i)},t^{(i)})$ with $H_i(y^{(i)},t^{(i)})x^{(i)}$ in place of $H_i(y^{(i)},t^{(i)})$. Suppose that $\det(DH_i(y^{(i)},t^{(i)})x^{(i)}) \ne 0$. Then, for $1\le j\le n_i$,*

$$\begin{aligned}&\partial_{x_j^{(i)}} g(H_i(y^{(i)},t^{(i)})x^{(i)})\\ &= \left\langle \nabla_{t^{(i)}} g(H_i(y^{(i)},t^{(i)})x^{(i)}), (DH_i(y^{(i)},t^{(i)})x^{(i)})^{-1}(\partial^L_{x_j^{(i)}}(H_i(y^{(i)},t^{(i)})x^{(i)}))\right\rangle,\end{aligned}$$

where $\nabla_{t^{(i)}} g = (\nabla_{t_1^{(i)}} g,\dots,\nabla_{t_{n_i}^{(i)}} g)$.

Lemma 6. *Suppose that* $\det(DH_i(y^{(i)},t^{(i)})x^{(i)})\neq 0$, *for* $i=1,2$. *Then*

$$\begin{aligned}&\partial_{x_i^{(1)}}\partial_{x_j^{(2)}}g(H_1(y^{(1)},t^{(1)})x^{(1)},H_2(y^{(2)},t^{(2)})x^{(2)})\\&\quad=\Big\langle\nabla_{t^{(1)}}\nabla_{t^{(2)}}g\big((DH_2(y^{(2)},t^{(2)})x^{(2)})^{-1}(\partial^L_{x_j^{(2)}}(H_2(y^{(2)},t^{(2)})x^{(2)}))\big),\\&\qquad\qquad(DH_1(y^{(1)},t^{(1)})x^{(1)})^{-1}(\partial^L_{x_i^{(1)}}(H_1(y^{(1)},t^{(1)})x^{(1)}))\Big\rangle,\end{aligned}$$

where $\nabla_{t^{(1)}}\nabla_{t^{(2)}}g$ *denotes the* $n_1\times n_2$ *matrix whose* (u,v) *component is*

$$\partial_{t_u^{(1)}}\partial_{t_v^{(2)}}g(H_1(y^{(1)},t^{(1)})x^{(1)},H_2(y^{(2)},t^{(2)})x^{(2)}).$$

Lemma 3 is from Lemma 7.1 of [25]. Lemma 4 follows from Lemma 3. See Lemma 7.2 of [25] for Lemmas 5 and 6.

To obtain (2.21), by Lemma 4 it suffices to prove that

$$\left|\iint gb(t^{(1)},t^{(2)})\prod_{i=1}^{2}\partial_{x_{j_i}^{(i)}}F_{j_i}^{(i)}(x^{(i)})\,\bar{d}t\,dx\right|\leq C\rho^{\delta\epsilon(k_1+k_2+c)}.\tag{2.22}$$

for all j_i, $1\leq j_i\leq n_i$, where g is as in (2.21),

$$b(t^{(1)},t^{(2)})=G_{22}\prod_{i=1}^{2}\prod_{u=1}^{n_1}\psi(t_u^{(i)}).$$

We can deduce (2.22) from the estimate

$$\left|\int\partial_{x_{j_1}^{(1)}}\partial_{x_{j_2}^{(2)}}gb(t^{(1)},t^{(2)})\,\bar{d}t\right|\leq C\rho^{-d\epsilon(k_1+k_2-c)}\tag{2.23}$$

for all $x\in B_1(0,C\rho^{k_1})\times B_2(0,C\rho^{k_2})$ with a sufficiently small $\epsilon>0$ and a constant $d>0$, if we apply integration by parts and use the L^1 norm estimate for $F_{j_i}^{(i)}$ in Lemma 4. By Lemma 6, we can replace $\partial_{x_{j_1}^{(1)}}\partial_{x_{j_2}^{(2)}}g$ by the expression involving $\nabla_{t^{(1)}}\nabla_{t^{(2)}}g$, and using integration by parts, we can get (2.23) from the estimate

$$\begin{aligned}\left|\int g\partial_{t_u^{(1)}}\partial_{t_v^{(2)}}\left[h_u^{(1)}(y^{(1)},t^{(1)},x^{(1)})h_v^{(2)}(y^{(2)},t^{(2)},x^{(2)})b(t^{(1)},t^{(2)})\right]\,\bar{d}t\right|&\\\leq C\rho^{-d\epsilon(k_1+k_2-c)},&\end{aligned}\tag{2.24}$$

where

$$(DH_i(y^{(i)},t^{(i)})x^{(i)})^{-1}(\partial^L_{x_{j_i}^{(i)}}x^{(i)})=(h_1^{(i)},\ldots,h_{n_i}^{(i)})$$

(note that $\partial^L_{x_{j_i}^{(i)}}(H_i(y^{(i)},t^{(i)})x^{(i)})=\partial^L_{x_{j_i}^{(i)}}x^{(i)}$). We see that

$$|\nabla_{t^{(1)}}\nabla_{t^{(2)}}b(t^{(1)},t^{(2)})|\leq C\rho^{-\epsilon(k_1+k_2)}\rho^c$$

and

$$|\det(DH_i(y^{(i)},t^{(i)})x^{(i)})|\geq C\rho^{k_i\epsilon}$$

on the support of b, since

$$\begin{aligned}|\det(DH_i(y^{(i)},t^{(i)})x^{(i)})| &= |\det C[x^{(i)}]\det DH_i(y^{(i)},t^{(i)})|\\ &\geq C|\det DH_i(y^{(i)},t^{(i)})|\end{aligned}$$

(see [25] and Section 2 of [21]). By these estimates along with Cramer's formula, we have the pointwise estimate

$$\left|\partial_{t_u^{(1)}}\partial_{t_v^{(2)}}\left[h_u^{(1)}(y^{(1)},t^{(1)},x^{(1)})h_v^{(2)}(y^{(2)},t^{(2)},x^{(2)})b(t^{(1)},t^{(2)})\right]\right| \leq C\rho^{-d\epsilon(k_1+k_2-c)},$$

which implies (2.24).

Case 3: $i = 1$ and $j = 2$. Using Lemmas 4 and 5 for $i = 2$ and arguing similarly to the proof of the case $i = 2$, $j = 2$, we have

$$\begin{aligned}&\left|\iint \Delta_{k_1,k_2}(x)gG_{12}\prod_{u=1}^{n_1}(F(y_u^{(1)},y_u^{(2)})\prod_{i=1}^{2}\psi^{(i)}(t_u^{(i)}))\,\bar{d}t^{(2)}\,dx\right|\\ &\leq C\rho^{\delta\epsilon(k_2+c)}\chi_{[0,1]}\left(\rho^{-k_1\epsilon}\left|\det(DH_1(y^{(1)},t^{(1)}))\right|\right)\prod_{u=1}^{n_1}(|F(y_u^{(1)},y_u^{(2)})|\psi(t_u^{(1)}))\end{aligned}$$

uniformly in $y^{(i)} \in (D_0^{(i)})^{n_1}$, $t^{(1)} \in [1,\rho^2]^{n_1}$ and $w_1,\ldots,w_n \in B_1(0,C\rho^2) \times B_2(0,C\rho^2)$. Integrating this with respect to $t^{(1)}$ and $y^{(i)}$ and using Hölder's inequality and (2.19), we have

$$\begin{aligned}&\left|\iint \Delta_{k_1,k_2}(x)gG_{12}\prod_{u=1}^{n_1}(F(y_u^{(1)},y_u^{(2)})\prod_{i=1}^{2}\psi(t_u^{(i)}))\,dy\,\bar{d}t\,dx\right|\\ &\qquad\leq C\rho^{\delta\epsilon(k_2+c)}\rho^{\delta\epsilon(k_1+c)/s'}\|F\|_s^{n_1}.\end{aligned}$$

This implies the desired estimate. If $i = 2$ and $j = 1$, (2.18) can be proved similarly.

If $k_1 \geq 0, k_2 \leq -1$, to show (2.13) we apply arguments similar to those for $k_1, k_2 \leq -1$ above. Let

$$G_j = \zeta_j\left(\rho^{-\epsilon k_2}\det(DH_2(y^{(2)},t^{(2)}))\right),$$

for $j = 1, 2$. Then, arguing as in the proof of (2.18) with respect to $y^{(2)}, t^{(2)}, x^{(2)}$ variables, we have

$$\begin{aligned}&\left|\iiint \Delta_{k_1,k_2}(x)gG_j\prod_{u=1}^{n_1}(F(y_u^{(1)},y_u^{(2)})\prod_{i=1}^{2}\psi(t_u^{(i)}))\,dy\,\bar{d}t\,dx\right|\\ &\qquad\leq C(\log\rho)^{2n_1}\rho^{\delta\epsilon(k_2+c)/s'}\|F\|_s^{n_1},\end{aligned}$$

for $j = 1, 2$, where the function g is as above. This and Lemma 2 imply (2.13). We can prove (2.14) similarly. This completes the proof of Lemma 1. □

3. Proof of Proposition 1.

By a standard method (see Lemma 6 of [21] for the one-parameter case) we can prove the following Littlewood-Paley inequalities.

Lemma 7. *Let $1 < p < \infty$ and $\Delta_{k_1,k_2} = \Delta^{(1)}_{k_1} \otimes \Delta^{(2)}_{k_2}$ (see Lemma 1). Then*

$$\left\| \sum_{k_1,k_2} f_{k_1,k_2} * \Delta_{k_1,k_2} \right\|_p \leq C_p \left\| \left(\sum_{k_1,k_2} |f_{k_1,k_2}|^2 \right)^{1/2} \right\|_p, \tag{3.1}$$

$$\left\| \left(\sum_{k_1,k_2} |f * \Delta_{k_1,k_2}|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p, \tag{3.2}$$

where the constant C_p is independent of $\rho \geq 2$.

Define

$$M_F f(x) = \sup_j |f * S_j(|F|)(x)|,$$

where we write $j = (j_1, j_2) \in \mathbb{Z} \times \mathbb{Z}$ and $S_j(F) = S_{j_1,j_2}(F)$ is as in Section 2 (see (2.1)). Similar notation will be used in what follows. Let $\mu^* f = M_F f$. We prove the following result for μ^* along with Proposition 1.

Lemma 8. *Let $p > 1, s \in (1, 2], \rho = 2^{s'}$ and $F \in L^s(D_0)$. Then, there exists a positive constant C_p independent of s and F such that*

$$\|\mu^* f\|_p \leq C_p (s-1)^{-2} \|F\|_s \|f\|_p.$$

Proof. Put $U_\sigma = U_\sigma(F)$ with $\rho = 2^{s'}$ (see (2.2)) and write $U_\sigma f = \sum_{k^{(1)},k^{(2)}} U_{k^{(1)},k^{(2)}} f$, where $k^{(i)} = (k^{(i)}_1, k^{(i)}_2) \in \mathbb{Z}^2$ and

$$U_{k^{(1)},k^{(2)}} f = \sum_j \sigma_j f * \Delta_{k^{(1)}+j} * \nu_j * \Delta_{k^{(2)}+j}, \quad \nu_j = \nu_j(F), \quad j = (j_1, j_2).$$

Fix $k^{(1)}, k^{(2)} \in \mathbb{Z}^2$. Using Lemma 1 with $\rho = 2^{s'}$ and duality, we have

$$\|f * \Delta_k * \nu_j\|_2 \leq C(s-1)^{-2} \|F\|_s \|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|k_i - j_i| - c)),$$

where $\lambda(t) = \min(1, 2^{-t})$. Applying this and Lemma 1, to ν_j and $\tilde{\nu_j}$, and noting that

$$\|\Delta_{k^{(2)}+j} * \Delta_{k^{(2)}+j'}\|_1 \leq C \prod_{i=1}^{2} \lambda(\epsilon(|j_i - j'_i| - c)), \quad j' = (j'_1, j'_2),$$

along with $\|\Delta_k\|_1 \leq C$, we have

$$\begin{aligned} &\left\| f * (\Delta_{k^{(1)}+j} * \nu_j) * (\Delta_{k^{(2)}+j} * \Delta_{k^{(2)}+j'}) * (\tilde{\nu}_{j'} * \Delta_{k^{(1)}+j'}) \right\|_2 \\ &\qquad \leq CA^2 \|f\|_2 \prod_{i=1}^{2} \lambda(2\epsilon(|k^{(1)}_i| - c)) \lambda(\epsilon(|j_i - j'_i| - c)), \end{aligned} \tag{3.3}$$

$$\left\| f * \Delta_{k^{(1)}+j} * (\nu_j * \Delta_{k^{(2)}+j}) * (\Delta_{k^{(2)}+j'} * \tilde{\nu}_{j'}) * \Delta_{k^{(1)}+j'} \right\|_2 \\ \leq CA^2 \|f\|_2 \prod_{i=1}^{2} \lambda(2\epsilon(|k_i^{(2)}| - c)), \quad (3.4)$$

where $A = (s-1)^{-2}\|F\|_s$. Taking the geometric mean, by (3.3) and (3.4) we have

$$\left\| f * \Delta_{k^{(1)}+j} * \nu_j * \Delta_{k^{(2)}+j} * \Delta_{k^{(2)}+j'} * \tilde{\nu}_{j'} * \Delta_{k^{(1)}+j'} \right\|_2 \\ \leq CA^2 \|f\|_2 \prod_{i=1}^{2} \left(\prod_{m=1}^{2} \lambda(\epsilon(|k_i^{(m)}| - c)) \right) \lambda(\epsilon(|j_i - j_i'| - c)/2).$$

We can treat

$$\|f * \Delta_{k^{(2)}+j'} * \tilde{\nu}_{j'} * \Delta_{k^{(1)}+j'} * \Delta_{k^{(1)}+j} * \nu_j * \Delta_{k^{(2)}+j}\|_2$$

similarly. Thus, by the Cotlar-Knapp-Stein lemma it follows that

$$\left\| U_{k^{(1)},k^{(2)}} f \right\|_2 \leq CA\|f\|_2 \prod_{m=1}^{2} \prod_{i=1}^{2} \lambda(\epsilon(|k_i^{(m)}| - c)/2) \quad (3.5)$$

uniformly in σ. From (3.5) we deduce that

$$\|U_\sigma f\|_2 \leq \sum_{k^{(1)},k^{(2)}} \|U_{k^{(1)},k^{(2)}} f\|_2 \leq CA\|f\|_2. \quad (3.6)$$

Let $\theta \in (0,1)$. Let $\{p_j\}_1^\infty$ be a sequence of positive numbers defined by $p_1 = 2$ and $1/p_{j+1} = 1/2 + (1-\theta)/(2p_j)$ for $j \geq 1$. Then, $1/p_j = (1-a^j)/(1+\theta)$, with $a = (1-\theta)/2$. We note that $\{p_j\}$ is decreasing and converges to $1+\theta$. We prove that

$$\|U_\sigma f\|_{p_m} \leq C_m A \, \|f\|_{p_m}, \quad m \geq 1, \quad (3.7)$$

for all $F \in L^s(D_0)$, where C_m is a constant independent of σ, F and s. This follows from (3.6) for $m = 1$. We fix $m \geq 1$ and assume (3.7) for this m. If it is applied to $U_\sigma(|F|)$, via the Khintchine inequality, we see that

$$\|g(f)\|_{p_m} \leq CA\|f\|_{p_m}, \quad (3.8)$$

where

$$g(f) = \left(\sum_j |f * \nu_j(|F|)|^2 \right)^{1/2}, \qquad j = (j_1, j_2).$$

Let $\nu^*(f) = \sup_j |f * |\nu_j||$, $\nu_j = \nu_j(F)$, and $\Phi^*(f) = \sup_j |f * \Phi_j(|F|)|$. Then

$$\begin{aligned} \nu^*(f) &\leq \mu^*(|f|) + C\Phi^*(|f|) + C\mu_1^*(|f|) + C\mu_2^*(|f|) \quad (3.9) \\ &\leq g(|f|) + C\Phi^*(|f|) + C\mu_1^*(|f|) + C\mu_2^*(|f|), \end{aligned}$$

where

$$\mu_i^* f = \sup_j |f * \Phi_j^{(i)}(|F|)|.$$

It is easy to see that

$$\Phi^*(|f|) \leq C(s-1)^{-2}\|F\|_1 Mf, \tag{3.10}$$

where M is the strong maximal function defined as

$$Mf(x^{(1)}, x^{(2)}) = \sup_{t_1, t_2 > 0} t_1^{-\gamma_1} t_2^{-\gamma_2} \int_{B_1(x^{(1)}, t_1) \times B_2(x^{(2)}, t_2)} |f(y^{(1)}, y^{(2)})|\, dy^{(1)}\, dy^{(2)}.$$

By Lemma 7 of [21] and the Hardy-Littlewood maximal theorem (see [7, 12]),

$$\|\mu_i^* f\|_p \leq C(s-1)^{-2}\|F\|_s \|f\|_p, \quad p > 1. \tag{3.11}$$

By the estimates (3.8)–(3.11) and L^p boundedness of M it follows that

$$\|\nu^*(f)\|_{p_m} \leq CA\|f\|_{p_m}. \tag{3.12}$$

By (3.12) and the estimate $\|\nu_j\|_1 \leq CA$ we have

$$\left\| \left(\sum |g_k * \nu_k|^2 \right)^{1/2} \right\|_{r_m} \leq CA \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_{r_m}, \tag{3.13}$$

where $1/r_m - 1/2 = 1/(2p_m)$ (see [10] and also [17, 18]). Thus, applying the Littlewood-Paley theory (Lemma 7) and (3.13) we have

$$\begin{aligned} \|U_{k^{(1)},k^{(2)}} f\|_{r_m} &\leq C \left\| \left(\sum_j |f * \Delta_{k^{(1)}+j} * \nu_j|^2 \right)^{1/2} \right\|_{r_m} \\ &\leq CA \left\| \left(\sum_j |f * \Delta_{k^{(1)}+j}|^2 \right)^{1/2} \right\|_{r_m} \\ &\leq CA\|f\|_{r_m}. \end{aligned} \tag{3.14}$$

Since $1/p_{m+1} = (1-\theta)/r_m + \theta/2$, interpolating between (3.5) and (3.14), we get

$$\|U_{k^{(1)},k^{(2)}} f\|_{p_{m+1}} \leq CA\|f\|_{p_{m+1}} \prod_{\ell=1}^{2} \prod_{i=1}^{2} \lambda(\theta\epsilon(|k_i^{(\ell)}| - c)/2), \tag{3.15}$$

and hence

$$\|U_\sigma f\|_{p_{m+1}} \leq \sum_{k^{(1)},k^{(2)}} \|U_{k^{(1)},k^{(2)}} f\|_{p_{m+1}} \leq CA\|f\|_{p_{m+1}},$$

which proves (3.7) for all m by induction. For any $p \in (1,2]$, we take $\theta \in (0,1)$ such that $p \in (1+\theta, 2]$ and define the sequence $\{p_j\}_1^\infty$ by using it. Then, we have $p_{j+1} < p \leq p_j$ with some j. It follows that

$$\|U_\sigma f\|_p \leq CA\|f\|_p \tag{3.16}$$

by interpolation between the estimates (3.7) with $m = j$ and $m = j+1$. By the estimate (3.16) we have $\|g(f)\|_p \leq CA\|f\|_p$ for $p \in (1,2]$, where $g(f)$ is as in (3.8). This estimate and (3.9), (3.10) with the strong maximal theorem, (3.11) imply Lemma 8 for $p \in (1,2]$.

If $p > 2$, interpolating between the estimate for $p = 2$ of Lemma 8 and the estimate

$$\|\mu^*(f)\|_\infty \le C(\log \rho)^2 \|F\|_1 \|f\|_\infty,$$

we get the desired result. This completes the proof of Lemma 8. □

Proof of Proposition 1. Since $Tf = U_\sigma(K_0)(f)$ if $\sigma_j = 1$ for all $j = (j_1, j_2)$, by (3.16) we have

$$\|Tf\|_p \le C(s-1)^{-2}\|\Omega\|_s \|f\|_p \quad \text{for } p \in (1,2];$$

a duality argument will imply the conclusion for $p \in [2, \infty)$. □

4. L^p estimates for certain maximal functions

We prove the following result, which may be useful in studying L^p boundedness of T_* in (1.5).

Proposition 2. *Let s, Ω be as in Proposition* 1*. We define*

$$R(f)(x) = \sup_{k_1,k_2 \in \mathbb{Z}} \left| \sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0(x) \right|,$$

where $S_{j_1,j_2}K_0$ is as in Section 2*. Let $A = (s-1)^{-2}\|\Omega\|_s$ and $\rho = 2^{s'}$. Then, for $p \in (1, \infty)$ we have*

$$\|R(f)\|_p \le C_p A \|f\|_p$$

with some positive constant C_p independent of $s \in (1,2]$ and $\Omega \in L^s$.

Proof. We write $\varphi_{k_1,k_2} = \sum_{m_1 \ge k_1+2, m_2 \ge k_2+2} \Delta_{m_1,m_2} = \varphi^{(1)}_{k_1} \otimes \varphi^{(2)}_{k_2}$, where $\Delta_{m_1,m_2} = \Delta^{(1)}_{m_1} \otimes \Delta^{(2)}_{m_2}$ and $\varphi^{(i)}_{k_i} = \delta^{(i)}_{\rho^{k_i+1}} \phi^{(i)}$. According to the decomposition

$$\begin{aligned}
\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0(x) &= T(f) * \varphi_{k_1,k_2} \\
&\quad - \left(\sum_{j_1=-\infty}^{k_1-1} \sum_{j_2=-\infty}^{\infty} f * S_{j_1,j_2} K_0 \right) * \varphi_{k_1,k_2} \\
&\quad - \left(\sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{k_2-1} f * S_{j_1,j_2} K_0 \right) * \varphi_{k_1,k_2} \\
&\quad + \left(\sum_{j_1=-\infty}^{k_1-1} \sum_{j_2=-\infty}^{k_2-1} f * S_{j_1,j_2} K_0 \right) * \varphi_{k_1,k_2} \\
&\quad + \left(\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0 \right) * (\delta - \varphi_{k_1,k_2}),
\end{aligned}$$

where δ is the delta function, we see that

$$R(f) \leq \sup_{k_1,k_2} |T(f) * \varphi_{k_1,k_2}| + M_1 + M_2 + M_3 + M_4 \tag{4.1}$$

with

$$M_1 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=-\infty}^{k_1-1} \sum_{j_2=-\infty}^{\infty} f * S_{j_1,j_2} K_0 \right) * \varphi_{k_1,k_2} \right|,$$

$$M_2 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{k_2-1} f * S_{j_1,j_2} K_0 \right) * \varphi_{k_1,k_2} \right|,$$

$$M_3 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=-\infty}^{k_1-1} \sum_{j_2=-\infty}^{k_2-1} f * S_{j_1,j_2} K_0 \right) * \varphi_{k_1,k_2} \right|,$$

$$M_4 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0 \right) * (\delta - \varphi_{k_1,k_2}) \right|.$$

Note that Proposition 1 and the strong maximal theorem imply that

$$\Big\| \sup_{k_1,k_2} |T(f) * \varphi_{k_1,k_2}| \Big\|_p \leq C\|M(Tf)\|_p \leq C_p A \|f\|_p, \quad p \in (1,\infty). \tag{4.2}$$

Thus, to prove Proposition 2, it remains to give the L^p estimates of M_i ($i = 1,2,3,4$) by (4.1).

Firstly, we consider M_4. Note that $M_4 \leq Q_1 + Q_2 + Q_3$, where

$$Q_1 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0 \right) * \left((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes \delta^{(2)} \right) \right|,$$

$$Q_2 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0 \right) * \left(\delta^{(1)} \otimes (\delta^{(2)} - \varphi^{(2)}_{k_2}) \right) \right|,$$

$$Q_3 = \sup_{k_1,k_2} \left| \left(\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2} K_0 \right) * \left((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes (\delta^{(2)} - \varphi^{(2)}_{k_2}) \right) \right|.$$

Now let us give the estimates of Q_1, Q_2 and Q_3, one by one.

Estimate of Q_3. We write $j = (j_1, j_2)$ and $k = (k_1, k_2)$, then it is easy to check that

$$Q_3 \leq \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} N_j(f),$$

where

$$N_j(f) = \sup_{k_1,k_2} \left| (f * S_{j+k} K_0) * \left((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes (\delta^{(2)} - \varphi^{(2)}_{k_2}) \right) \right|.$$

Hence

$$\|Q_3\|_p \leq \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \|N_j(f)\|_p.$$

Since Lemma 8 and the Hardy-Littlewood maximal theorem imply that

$$\|N_j(f)\|_u \leq C_u A\|f\|_u \quad \text{for } u > 1, \tag{4.3}$$

if we can show that there are $\delta, c > 0$ such that

$$\|N_j(f)\|_2 \leq CA\|f\|_2 \prod_{i=1}^{2} \lambda(\delta(j_i - c)), \tag{4.4}$$

then we may get

$$\|N_j(f)\|_p \leq CA\|f\|_p \prod_{i=1}^{2} \lambda(\theta\delta(j_i - c))$$

for some $\theta \in (0, 1]$ by interpolating between (4.3) and (4.4). In fact, for $p \in (1, \infty)$, it is enough to take $u \in (1, \infty)$ and $\theta \in (0, 1]$ such that $1/p = (1 - \theta)/u + \theta/2$. Thus we have

$$\|Q_3\|_p \leq \sum_j \|N_j(f)\|_p \leq CA\|f\|_p. \tag{4.5}$$

So, it suffices to verify (4.4) for estimating L^p norm of Q_3.

Let $J_k = (\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes (\delta^{(2)} - \varphi^{(2)}_{k_2})$. Then

$$N_j(f) \leq \left(\sum_k \left|f * S_{j+k}K_0 * J_k\right|^2\right)^{1/2}. \tag{4.6}$$

Fix j and let

$$V_\sigma f = \sum_k \sigma_k f * S_{j+k}K_0 * J_k,$$

where $\sigma = \{\sigma_k\}$, $\sigma_k = 1$ or -1. If we can show that

$$\|V_\sigma f\|_2 \leq CA\|f\|_2 \prod_{i=1}^{2} \lambda(\delta(j_i - c)) \tag{4.7}$$

for some $\delta, c > 0$, uniformly in σ, then (4.4) follows from (4.6), (4.7) and Khintchine's inequality.

To prove (4.7) we argue similarly to the proof of (3.5). By the Cotlar-Knapp-Stein lemma, it is easy to see that (4.7) can be deduced from the following two estimates:

$$\begin{aligned}&\|f * S_{j+k^{(1)}}K_0 * J_{k^{(1)}} * J_{k^{(2)}} * S_{j+k^{(2)}}\tilde{K_0}\|_2 \\ &\qquad \leq CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\delta(j_i - c))\lambda(\delta(|k_i^{(1)} - k_i^{(2)}| - c)) \end{aligned}\tag{4.8}$$

and

$$\|f * J_{k^{(2)}} * S_{j+k^{(2)}}\tilde{K_0} * S_{j+k^{(1)}}K_0 * J_{k^{(1)}}\|_2 \leq CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\delta(j_i - c))\lambda(\delta(|k_i^{(1)} - k_i^{(2)}| - c)) \quad (4.9)$$

for some $\delta, c > 0$. Hence, to estimate Q_3 it remains to give the estimates (4.8) and (4.9).

Proof of (4.8). Note that $\delta^{(i)} - \varphi_{k_i}^{(i)} = \sum_{m_i \leq k_i+1} \Delta_{m_i}^{(i)}$, $J_k = \sum_{m \leq k+(1,1)} \Delta_m$, where $k = (k_1, k_2)$, $m = (m_1, m_2)$ and $m \leq k$ means that $m_1 \leq k_1$ and $m_2 \leq k_2$. Therefore,

$$\|f * S_{j+k^{(1)}}K_0 * J_{k^{(1)}} * J_{k^{(2)}} * S_{j+k^{(2)}}\tilde{K_0}\|_2 \leq \sum_{\substack{m^{(1)} \leq k^{(1)}+(1,1),\\ m^{(2)} \leq k^{(2)}+(1,1)}} \|f * S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}} * \Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0}\|_2. \quad (4.10)$$

Lemma 1 implies that

$$\|f * (S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}}) * (\Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0})\|_2 \leq CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|j_i + k_i^{(1)} - m_i^{(1)}| - c))\lambda(\epsilon(|j_i + k_i^{(2)} - m_i^{(2)}| - c)). \quad (4.11)$$

Also, we see that

$$\|f * S_{j+k^{(1)}}K_0 * (\Delta_{m^{(1)}} * \Delta_{m^{(2)}}) * S_{j+k^{(2)}}\tilde{K_0}\|_2 \leq CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|m_i^{(1)} - m_i^{(2)}| - c)). \quad (4.12)$$

By the estimates (4.11) and (4.12) it follows that

$$\begin{aligned}\|f * S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}} * \Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0}\|_2 &\leq CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|j_i + k_i^{(1)} - m_i^{(1)}| - c)/2)\lambda(\epsilon(|j_i + k_i^{(2)} - m_i^{(2)}| - c)/2) \\ &\quad \times \lambda(\epsilon(|m_i^{(1)} - m_i^{(2)}| - c)/2). \quad (4.13)\end{aligned}$$

By (4.10) and (4.13) we obtain (4.8).

The proof of the estimate (4.9) is similar. In fact,

$$\begin{aligned}
&\|f * J_{k^{(2)}} * S_{j+k^{(2)}}\tilde{K_0} * S_{j+k^{(1)}}K_0 * J_k^{(1)}\|_2 \qquad (4.14)\\
&\le \sum_{\substack{m^{(1)} \le k^{(1)}+(1,1),\\ m^{(2)} \le k^{(2)}+(1,1)}} \|f * \Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0} * S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}}\|_2 \\
&\le \sum_{\substack{m^{(1)} \le k^{(1)}+(1,1),\\ m^{(2)} \le k^{(2)}+(1,1)}} \sum_{\ell^{(1)},\ell^{(2)}} \|f * \Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0} * \Delta_{\ell^{(1)}} * \Delta_{\ell^{(2)}} * S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}}\|_2.
\end{aligned}$$

Lemma 1 implies

$$\begin{aligned}
&\|f * (\Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0}) * (\Delta_{\ell^{(1)}} * \Delta_{\ell^{(2)}}) * (S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}})\|_2 \\
&\quad \le CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|j_i + k_i^{(2)} - m_i^{(2)}| - c))\lambda(\epsilon(|j_i + k_i^{(1)} - m_i^{(1)}| - c)) \\
&\qquad\qquad \times \lambda(\epsilon(|\ell_i^{(1)} - \ell_i^{(2)}| - c)), \qquad (4.15)
\end{aligned}$$

$$\begin{aligned}
&\|f * \Delta_{m^{(2)}} * (S_{j+k^{(2)}}\tilde{K_0} * \Delta_{\ell^{(1)}}) * (\Delta_{\ell^{(2)}} * S_{j+k^{(1)}}K_0) * \Delta_{m^{(1)}}\|_2 \\
&\le CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|j_i + k_i^{(2)} - \ell_i^{(1)}| - c))\lambda(\epsilon(|j_i + k_i^{(1)} - \ell_i^{(2)}| - c)). \qquad (4.16)
\end{aligned}$$

By (4.15) and (4.16), taking the geometric mean,

$$\begin{aligned}
&\|f * \Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0} * \Delta_{\ell^{(1)}} * \Delta_{\ell^{(2)}} * S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}}\|_2 \\
&\le CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|j_i + k_i^{(2)} - m_i^{(2)}| - c)/2)\lambda(\epsilon(|j_i + k_i^{(1)} - m_i^{(1)}| - c)/2) \\
&\qquad \times \lambda(\epsilon(|\ell_i^{(1)} - \ell_i^{(2)}| - c)/2)\lambda(\epsilon(|j_i + k_i^{(2)} - \ell_i^{(1)}| - c)/2) \\
&\qquad\qquad \times \lambda(\epsilon(|j_i + k_i^{(1)} - \ell_i^{(2)}| - c)/2). \qquad (4.17)
\end{aligned}$$

Summing in $\ell^{(1)}$, $\ell^{(2)}$ in (4.17), we have

$$\begin{aligned}
&\|f * \Delta_{m^{(2)}} * S_{j+k^{(2)}}\tilde{K_0} * S_{j+k^{(1)}}K_0 * \Delta_{m^{(1)}}\|_2 \\
&\qquad \le CA^2\|f\|_2 \prod_{i=1}^{2} \lambda(\delta(|k_i^{(1)} - k_i^{(2)}| - c)) \\
&\quad \times \lambda(\epsilon(|j_i + k_i^{(2)} - m_i^{(2)}| - c)/2)\lambda(\epsilon(|j_i + k_i^{(1)} - m_i^{(1)}| - c)/2) \qquad (4.18)
\end{aligned}$$

for some $\delta, c > 0$. By (4.14) and (4.18) we have (4.9).

Estimate of Q_1. We decompose

$$\left(\sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2}K_0\right) * \left((\delta^{(1)} - \varphi_{k_1}^{(1)}) \otimes \delta^{(2)}\right) = W_1(k) - W_2(k) + W_3(k),$$

where

$$W_1(k) = \left(\sum_{j_1=k_1}^{\infty}\sum_{j_2=-\infty}^{\infty} f * S_{j_1,j_2}K_0\right) * \left((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes \varphi^{(2)}_{k_2}\right),$$

$$W_2(k) = \left(\sum_{j_1=k_1}^{\infty}\sum_{j_2=-\infty}^{k_2-1} f * S_{j_1,j_2}K_0\right) * \left((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes \varphi^{(2)}_{k_2}\right),$$

$$W_3(k) = \left(\sum_{j_1=k_1}^{\infty}\sum_{j_2=k_2}^{\infty} f * S_{j_1,j_2}K_0\right) * \left((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes (\delta^{(2)} - \varphi^{(2)}_{k_2})\right).$$

We note that $\sup_k |W_3(k)| = Q_3$. It is easy to see that

$$\sup_k |W_1(k)| \le \sup_{k_2}\left|\left(\sum_{j_1\ge 0} N_{j_1}(f)\right) * (\delta^{(1)} \otimes \varphi^{(2)}_{k_2})\right|,$$

where

$$N_{j_1}(f) = \sup_{k_1}\left|\sum_{j_2}(f * S_{k_1+j_1,j_2}K_0) * ((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes \delta^{(2)})\right|.$$

We need to prove

$$\|N_{j_1}(f)\|_2 \le CA\lambda(\delta(j_1 - c))\|f\|_2. \tag{4.19}$$

Thus, by using (4.19) and by applying interpolation suitably, we have

$$\|N_{j_1}(f)\|_p \le CA\lambda(\delta(j_1 - c))\|f\|_p$$

for some $\delta, c > 0$, and hence

$$\|\sup_k |W_1(k)|\|_p \le C\sum_{j_1}\|N_{j_1}(f)\|_p \le CA\|f\|_p. \tag{4.20}$$

To prove (4.19), let

$$V'_\sigma(f) = \sum_k \sigma_{k_1}(f * S_{(j_1,0)+k}K_0) * ((\delta^{(1)} - \varphi^{(1)}_{k_1}) \otimes \delta^{(2)}), \quad k = (k_1, k_2),$$

where $\sigma = \{\sigma_{k_1}\}$ is a sequence such that $\sigma_{k_1} = 1$ or $\sigma_{k_1} = -1$. Then as in the case of $N_j(f)$, it suffices to show

$$\|V'_\sigma(f)\|_2 \le CA\lambda(\delta(j_1 - c))\|f\|_2.$$

To prove this, as in the proof of (4.7) above, we need to estimate

$$\sum_{\substack{m_1^{(1)} \le k_1^{(1)}+1,\\ m_1^{(2)} \le k_1^{(2)}+1}} \|f * S_{(j_1,0)+k^{(1)}}K_0 * \Delta_{m^{(1)}} * \Delta_{m^{(2)}} * S_{(j_1,0)+k^{(2)}}\tilde{K_0}\|_2$$

and

$$\sum_{\substack{m_1^{(1)}\le k_1^{(1)}+1,\\ m_1^{(2)}\le k_1^{(2)}+1}} \|f*\Delta_{m^{(2)}}*S_{(j_1,0)+k^{(2)}}\tilde{K_0}*S_{(j_1,0)+k^{(1)}}K_0*\Delta_{m^{(1)}}\|_2.$$

However, it is easy to see that a computation similar to the one in the proof of (4.7) gives the desired results.

As for the estimate of $\sup_k |W_2(k)|$, we see that

$$\sup_k |W_2(k)| \le \sum_{j_1\ge 0}\sum_{j_2\ge 1} P_j,$$

where $j=(j_1,j_2)$ and

$$P_j = \sup_{k_1,k_2}\left|(f*S_{j_1+k_1,k_2-j_2}K_0)*\left((\delta^{(1)}-\varphi_{k_1}^{(1)})\otimes\varphi_{k_2}^{(2)}\right)\right|.$$

As in the arguments above, we need to show that

$$\|P_{j_1,j_2}\|_2 \le CA\|f\|_2\prod_{i=1}^{2}\lambda(\delta(j_i-c)) \tag{4.21}$$

for some $\delta, c>0$. Let $j_*=(j_1,-j_2)$. To prove this we use the following estimates:

$$\begin{aligned}
&\|f*S_{j_*+k^{(1)}}K_0*\Delta_{m^{(1)}}*\Delta_{m^{(2)}}*S_{j_*+k^{(2)}}\tilde{K_0}\|_2\\
&\qquad\le CA^2\|f\|_2\prod_{i=1}^{2}\lambda(\epsilon(|(j_*)_i+k_i^{(1)}-m_i^{(1)}|-c)/2)\\
&\qquad\quad\times\lambda(\epsilon(|(j_*)_i+k_i^{(2)}-m_i^{(2)}|-c)/2)\lambda(\epsilon(|m_i^{(1)}-m_i^{(2)}|-c)/2)
\end{aligned}$$

and

$$\begin{aligned}
&\|f*\Delta_{m^{(2)}}*S_{j_*+k^{(2)}}\tilde{K_0}*S_{j_*+k^{(1)}}K_0*\Delta_{m^{(1)}}\|_2\\
&\qquad\le CA^2\|f\|_2\prod_{i=1}^{2}\lambda(\delta(|k_i^{(1)}-k_i^{(2)}|-c))\\
&\qquad\quad\times\lambda(\epsilon(|(j_*)_i+k_i^{(2)}-m_i^{(2)}|-c)/2)\lambda(\epsilon(|(j_*)_i+k_i^{(1)}-m_i^{(1)}|-c)/2)
\end{aligned}$$

for $m^{(1)}, m^{(2)}$ with $m_1^{(1)}\le k_1^{(1)}+1, m_1^{(2)}\le k_1^{(2)}+1$ and $m_2^{(1)}\ge k_2^{(1)}+2, m_2^{(2)}\ge k_2^{(2)}+2$. These estimates can be proved as (4.13) and (4.18). Using them as in the proof of (4.4), we can deduce that

$$\begin{aligned}
&\left\|\left(\sum_k\left|(f*S_{j_*+k}K_0)*\left((\delta^{(1)}-\varphi_{k_1}^{(1)})\otimes\varphi_{k_2}^{(2)}\right)\right|^2\right)^{1/2}\right\|_2\\
&\qquad\qquad\le CA\|f\|_2\prod_{i=1}^{2}\lambda(\delta(j_i-c))
\end{aligned}$$

for some $\delta, c>0$, which implies (4.21).

Interpolating between the estimates of (4.21) and Lemma 8, we have

$$\|P_{j_1,j_2}(f)\|_p \le CA\|f\|_p \prod_{i=1}^{2} \lambda(\delta(j_i - c))$$

for some $\delta, c > 0$, and hence

$$\|\sup_k |W_2(k)|\|_p \le C \sum_{j_1,j_2} \|P_{j_1,j_2}(f)\|_p \le CA\|f\|_p. \tag{4.22}$$

By (4.5), (4.20) and (4.22)

$$\|Q_i\|_p \le CA\|f\|_p \tag{4.23}$$

for $i = 1$. We can prove (4.23) for $i = 2$ in the same way. From (4.5) and (4.23) for $i = 1, 2$, it follows that

$$\|M_4\|_p \le CA\|f\|_p. \tag{4.24}$$

Secondly, we give the estimate of M_3. Note that

$$M_3 \le \sum_{j_1 \ge 1} \sum_{j_2 \ge 1} Q_{j_1,j_2},$$

where

$$Q_{j_1,j_2} = \sup_{k_1,k_2} \left| (f * S_{k_1-j_1,k_2-j_2} K_0) * \left(\varphi^{(1)}_{k_1} \otimes \varphi^{(2)}_{k_2} \right) \right|.$$

Writing $\varphi^{(i)}_{k_i} = \sum_{m_i \ge k_i+2} \Delta^{(i)}_{m_i}$ and arguing as in the proof of (4.4), we can prove

$$\|Q_{j_1,j_2}\|_2 \le CA\|f\|_2 \prod_{i=1}^{2} \lambda(\delta(j_i - c))$$

for some $\delta, c > 0$, which implies, in the same way as above,

$$\|M_3\|_p \le CA\|f\|_p, \quad 1 < p < \infty. \tag{4.25}$$

Alternatively, we can handle M_3 by noting $M_3 \le CAMf$ (see Lemma 10 of [21]).

Finally, let us estimate M_1 and M_2. We observe that

$$M_1 \le \sup_{k_2} \left| \left(\sum_{j_1 \ge 1} N'_{j_1}(f) \right) * (\delta^{(1)} \otimes \varphi^{(2)}_{k_2}) \right|,$$

where

$$N'_{j_1}(f) = \sup_{k_1} \left| \sum_{j_2} (f * S_{k_1-j_1,j_2} K_0) * (\varphi^{(1)}_{k_1} \otimes \delta^{(2)}) \right|.$$

In the same way as in the estimate of $N_{j_1}(f)$, we can prove

$$\|N'_{j_1}(f)\|_2 \le CA\lambda(\delta(j_1 - c))\|f\|_2.$$

Using this, we can show

$$\|M_i\|_p \le CA\|f\|_p, \quad 1 < p < \infty. \tag{4.26}$$

for $i = 1$. The estimate (4.26) for M_2 can be proved similarly.

By (4.1), (4.2), (4.24), (4.25) and (4.26) for $i = 1$ and 2, we obtain the conclusion of Proposition 2. $\square$

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